

# Configurations

## The Complexity of Card Games and Chess

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### 1 Introduction

Combinatorics is the study of finite or countable discrete structures. It focuses on counting and finding bounds in order to solve problems. It is used heavily in the computer science world in order to analyze the complexity of algorithms. One of the most studied subfields of combinatorics is Graph Theory. Graph Theory takes data points and names them nodes and when two nodes share a relation they are joined by an edge. Extensive research goes into finding the shortest path through these graphs and optimizing the connection of edges. In the past few years, Graph Theory became a huge topic of study because it is the math behind large computer networks.

This paper deals with the complexity behind analyzing the different configurations of a chessboard. Section 2 uses a famous combinatorial example of a deck of cards to show how difficult it is to find all possible enumerations of a set even when the size of the set does not seem to large. Section 3.2 shows a possible solution to the chessboard configuration problem, and the future of the project is addressed in the Conclusion, Section 4.

### 2 Deck of Cards

#### 2.1 Truly Random Shuffles

A standard deck of cards consists of 52 playing cards and 2 jokers. In order to use the cards for card games one removes the two jokers. Therefore, the number of possible orderings of the playable deck can be found by picking one card out of 52, then one more out of 51, and so on. This number can be represented as  $52!$  or:

80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000,000

or,  $8.0658175 \times 10^{67}$ . In order to properly get a random shuffle, one must perform a riffle shuffle 7 times. After 7 times, the cards have been misplaced enough to be able to get every possible combination after the last shuffle.

## 2.2 Will we reach the end?

Although the number of combinations is huge, there is still a finite number of orderings, meaning it is theoretically possible to find every single combination. Currently, there are 7.125 billion people in the world. If each person shuffled a deck of cards truly randomly every second, it would take  $3.5995515 \times 10^{50}$  years to see every permutation. However, the age of the universe is estimated to be 13.82 billion years, or  $1.382 \times 10^{10}$  making this number seem infinitesimal compared to time as humans can comprehend.

The fifth fastest super computer in the world at Argonne National Laboratory operates at a speed of 8.59 petaflops and the fastest super computer in the world runs at 33.86 petaflops. A petaflop is equivalent to one thousand million operations per second,  $1 \times 10^{15}$ . The combined speed of these two along with the rest of the top 5 is 85 petaflops. If the only job of these computers was to run through the possible orderings, it would still take  $3.01 \times 10^{43}$  years to complete. It is safe to say, that all possible combinations will not be found until there are major advances in computing machinery or perhaps never at all.

## 3 Chess

As seen in Section 2 even a seemingly simple concept of shuffling cards results in numbers still beyond human computation. Another famous game throughout history is the game of chess. The game involves two rival armies (white and black) squaring off along an  $8 \times 8$  collection of squares of alternating colors. The different pieces that make up each rival army have different moves allowing for different strategies to be found to trap the opponent's king.

### 3.1 Possible Board Configurations

A chessboard is an  $8 \times 8$  set of squares making 64 individual locations. Each team has 1 King, 1 Queen, 2 Bishops, 2 Knights, 2 Rooks, and 8 Pawns. This means there are 16 pieces per team and 32 pieces spread across the board. Pieces can be eliminated reducing this count, but there will never be less than 2 pieces on the board at a time because the king cannot be killed, only trapped. We can find the upper bound of the board layouts by picking 32 squares to place pieces on, then 31 squares, and so on until we have reached two. This method allows us to account for pieces that are captured when their square is landed on by a rival piece. Therefore, the upper bound of the number of possible layouts of a chessboard:

$$\sum_{k=2}^{32} \binom{64}{k} = \sum_2^{32} \frac{64!}{k!(64-k!)} = 1.013968 \times 10^{19} \quad (1)$$

However, this gives us the ordering for 32 identical pieces.

If we count each piece as being distinct we get:

$$\sum_{k=2}^{32} k! \binom{64}{k} = \sum_2^{32} k! \frac{64!}{k!(64-k!)} = 4.97275 \times 10^{53} \quad (2)$$

We use  $k!$  because after choosing  $k$  positions of "identical pieces" there are  $k!$  ways to arrange the  $k$  distinct pieces on the board. See Section 2.1 for another example of this kind of ordering.

In 1950, Claude Shannon estimated the possible number of chess positions to be roughly  $10^{43}$ . He calculated this using the equation:

$$\frac{64!}{32! \times 8!^2 \times 2!^6}$$

This solution discounts captures (only creates the valid combinations for 32 pieces) and promotions. However, it does take into account the 4 pieces on each side that have multiples: 8 Pawns, 2 Bishops, 2 Knights, and 2 Rooks per side) and treats the individual pieces within each set as the same. More recently, Victor Allis calculated the upper bound to be  $5 \times 10^{52}$  which is just one order of magnitude less than the upper bound calculation above. A modern mathematician James Grime has estimated that there are  $10^{40}$  sensible games of Chess that can be played.

Currently, the actual value of possible combinations is still unknown. When you consider only valid game positions (positions that obey all of the rules), the number is actually less than the upper bound from Equation 2. There are a few reasons for this:

1. Pawns are only capable of moving forward from their starting position, and can only go diagonal to remove an enemy game piece. Therefore, pawns cannot ever reach the row behind which they were placed. So, a pawns board is actually a  $7 \times 8$  arrangement of squares.
2. Bishops also pose a problem when computing the total. Each bishop can only travel on its assigned color, limiting the number of squares it can travel to to  $\frac{n}{2}$ . For each team (white and black) each bishop's set of possible locations is disjoint. So, the bishops on each side that start on the same color can be calculated in the same binomial coefficient by choosing 2 out of  $\frac{n}{2}$  possible squares.
3. Another variable that must be accounted for is promotions. If a pawn reaches the opposing side, it is promoted to a higher piece of the same color. While normally the player elects to promote to a queen, there are situations where a knight would allow the player to win instead of a queen. However, all possible substitutions would have to be counted to get an accurate number.
4. Finally, the issue arises that the upper bound allows for cases where both kings are in check which is an illegal board configuration.

The rest of the other pieces are capable of reaching every other position on the board. The queen can move any direction in a straight line and also along diagonals allowing her to quickly traverse the board. Rooks are able to move in straight lines only, but because a chessboard is made up of rows and columns, rooks can reach every possible location on the board by combining a vertical movement (forward or backwards with regard to the starting side) and a horizontal movement (left and right across a row). The two kings can also reach every square on the board. While this is a little harder to see, each king can move one

in both the horizontal, vertical and diagonal directions. Eventually, with enough turns the king, like the queen, is able to reach every square.

The hardest piece to visualize reaching every square is the knight. The knight moves in an "L" shape, moving two squares vertical and one square horizontal, or two squares horizontal and one square vertical (Figure 1). While it seems like each knight's set of valid locations should be less than 64, the famous computer science and math challenge problem "The Knight's Tour" shows that a knight can traverse the entire set of squares. The earliest recorded complete path was in 840AD by al-Asli ar-Rumi in a book about an ancient form of chess in the Middle East. There are a number of different paths the knight can take from many different starting positions creating a connected graph as shown in Figure 2.

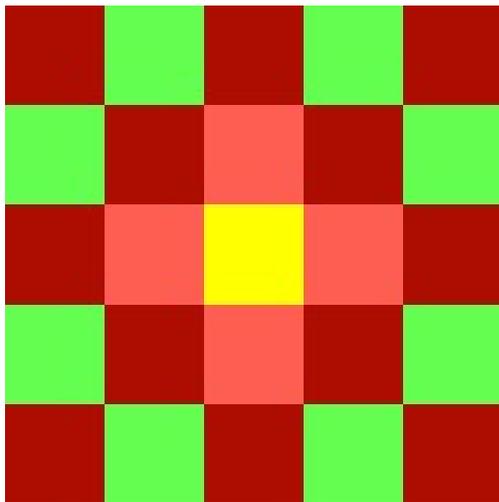


Figure 1: The possible ending locations for a knight at the yellow square.

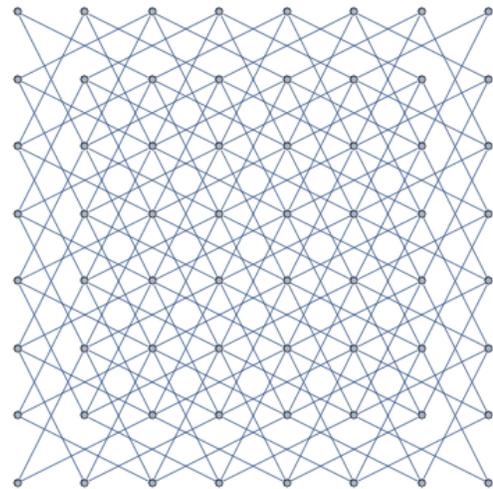


Figure 2: A graph showing a path of a knight through all vertices of the chessboard. (Wikipedia)

Therefore, I propose a new equation for calculating the possible legal calculations. First, we reduce the possible number of locations for individual pawns to  $8 \times 7$  or 56 locations. We also can reduce the possible locations of the bishops by combining the bishops of each colored square together (e.g. the black and white bishops that start on a white square are not affected by the choices of the other two bishops starting on a black square). So, the estimate equation can be represented as:

$$\sum_{k=0}^{16} \sum_{j=2}^{12} \sum_{m=0}^2 \sum_{n=0}^2 \binom{32}{m} \binom{32}{n} \binom{56 - m - n}{k} \binom{64 - k - m - n}{j} \quad (3)$$

$k$ : The number of pawns on the board.

$j$ : The number of pieces on the board not counting pawns and bishops.

$m$ : The number of bishops on the board beginning on white spaces.

$n$ : The number of bishops on the board beginning on black spaces.

Unfortunately, although this will be much closer to the actual answer, this expression still does not count promotions or the issue of the two kings in check. So, I would like to suggest a second approach to the problem.

### 3.2 Chess as a Graph

Another approach to solving this problem is to use graph theory. A chessboard can be represented as a collection of nodes and edges. Each edge represents the possible movement by any piece from the target node. Figure 3 shows the possible target squares in green and the impossible squares in red. While this is only a  $7 \times 7$  subset of the full chessboard, the 8 green squares on the edge of the picture (4 straight lines and 4 vertical lines) continue on until reaching the edges of the board. The graph representation of Figure 3 can be found in Figure 4.

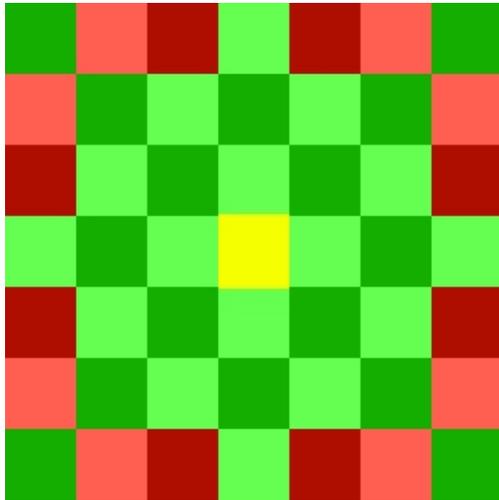


Figure 3: Valid target squares from the yellow source square on a  $7 \times 7$  subset of a chessboard.

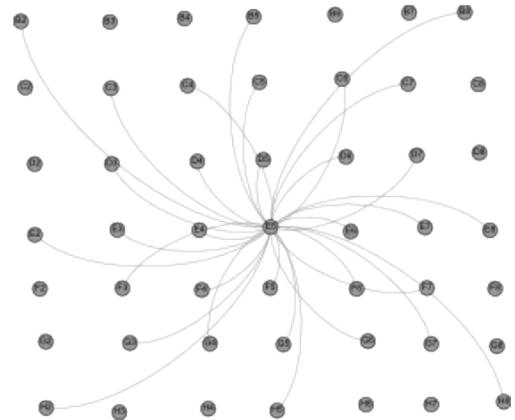


Figure 4: Graph representation of Figure 3.

The graph in Figure 5 is the graph of a standard  $8 \times 8$  chessboard. Each node has connections similar to the node in Figure 4. This graph may provide a easier way to look at the problem of finding all configurations. By choosing two colors (one color and uncolored counts as two different colors in this case) the edges of the graph can be colored. For the purposes of the paper, green will be our valid color and grey or no color will be used interchangeably to represent invalid edges. The valid edges are those edges that, at each turn of any game, a piece can move along. Figure 6 shows the colored edges for the first turn of a chess game.

The reason I believe this approach may work, is because graphs represent a physical form of the problem. There are actual edges and rules that must be followed. Of course, it is still too complex to count every possible combination of edges, there are over  $2^{800}$  possibilities. But I believe through experimentation and deduction, there is a formula that will allow the

valid arrangements of edges to be isolated. The number of valid colorings of the graph in Figure 5 will be the number of board configurations that are allowed.

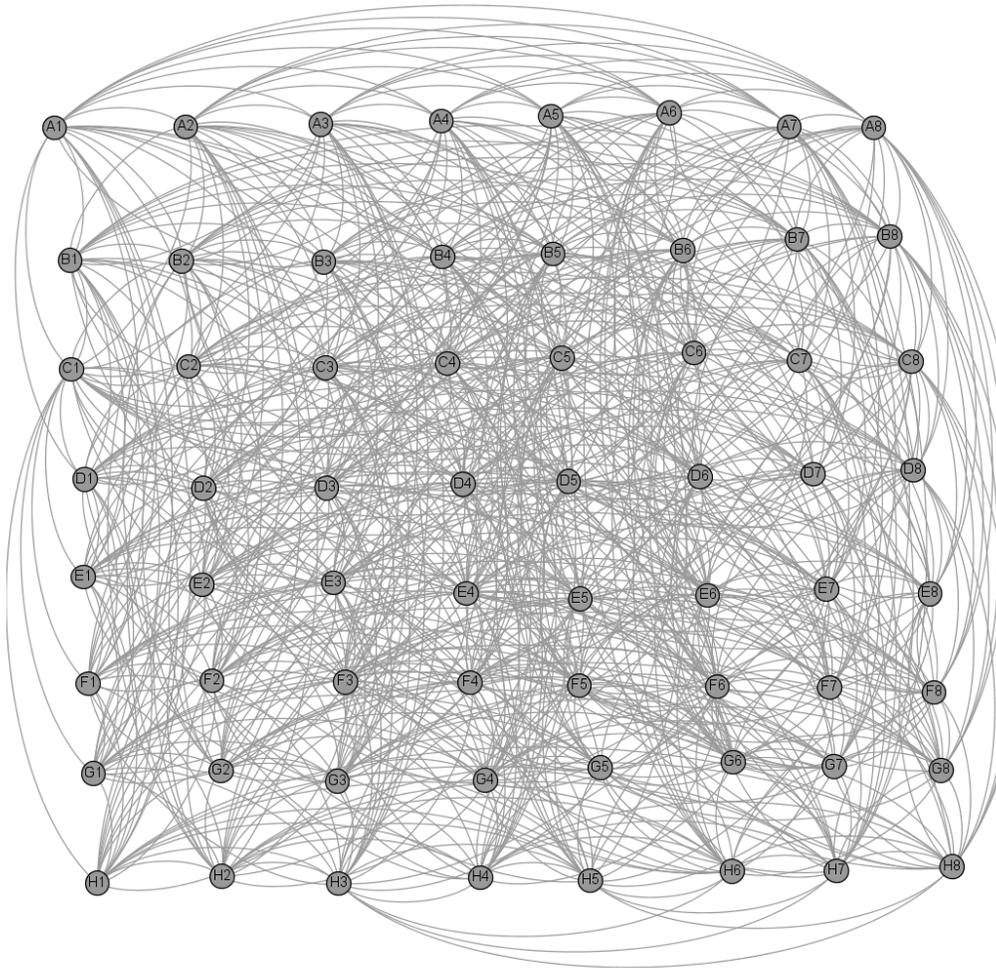


Figure 5: The full graph of a  $8 \times 8$  chessboard.

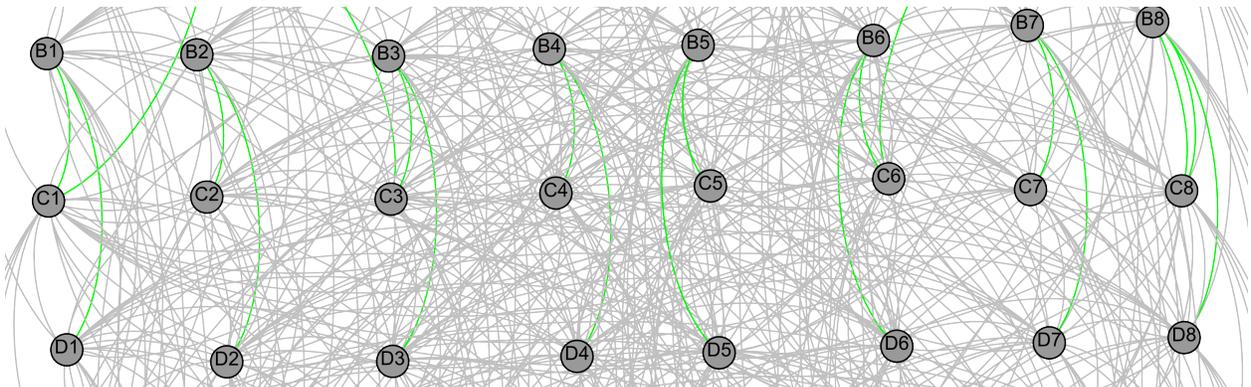


Figure 6: A section of the graph showing the coloring for the first move of a game.

## 4 Conclusion

As shown in the example with the shuffling cards problem, even simple configuration sets can be seemingly impossible to accurately depict. With the chess problem, there are a lot of expressions that get close, but still no one has been able to find an answer to the number of configurations. I truly believe that the graph may help minimize the complexity of the problem. If anything, the colorings of the graph can be used to count the number of times the 4th or 5th condition that Expression 1 failed on. Once those numbers are known, they can be subtracted from Expression 3 and the actual final answer can be found.

In order to solve this, either completely through the colorings of the graph, or using the colorings to fix Expression 3, the next step of the process will be to compute Expression 3. Currently, the expression is too complex for even Wolfram Alpha to handle. A simple computer program can be written and left to run in order to compute the sum. Secondly, a computer program can be written to better represent the different colorings of the graph of a chessboard. Once that is done, there will be a chance to find patterns and reduce the complexity perhaps even to a level where an exact number can be found.

## 5 Bibliography

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\*Unless specified all images were created by Matthew Treflek.\*